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ANALYSIS OF HEAT TRANSFER, INCLUDING AXIAL FLUID CONDUCTION, FOR LAMINAR TUBE FLOW WITH ARBITRARY CIRCUMFERENTIAL WALL HEAT FLUX VARIATION

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Abstract—An analysis is presented for the exact solution to the problem of thermal-entry-region heat transfer in a circular tube with an arbitrary circumferential wall heat flux, including the effect of axial fluid conduction. The solution is expressed in a power series form; the first 12 eigenvalues and eigenfunctions were obtained numerically. The least squares method was employed to determine the series expansion coefficients for the non-orthogonal system. A simple result is given for heat flux variation in the form $q = q_{av}(1+\cos\phi)$ which illustrates the simultaneous influence of circumferential wall heat flux variation and axial fluid conduction on wall temperature and Nusselt number.

NOMENCLATURE

a_n, b_n,	Fourier coefficients;
a_{n0}, a_{np}, b_{np},	expansion coefficients;
$\hat{a}_{n0}, \hat{a}_{np}$,	defined as $\hat{a}_{n0} = 4a_{n0}, \hat{a}_{np} = \frac{4p}{b}a_{np}$;
b,	heat flux parameter for the special example;
E,	error between the function and its power series expansion;
h,	heat-transfer coefficients;
p,	integer parameter;
$q(\phi)$,	arbitrary variation of circumferential wall heat flux;
$\bar{q}$,	defined by equation (3d);
$R_{np}(r^+)$,	eigenfunctions of the characteristic equation (12);
r,	radial coordinate;
r_0,	pipe radius;
$t(x, r, \phi)$,	local fluid temperature;
v,	average fluid velocity;
x,	axial coordinate, distance from pipe entrance.

Greek symbols

α,	thermal diffusivity;
θ,	local dimensionless fluid temperature;
θ^+,	dimensionless entrance region temperature;
θ_{fd},	asymptotic dimensionless fluid temperature;
$\hat{\theta}_{fd}$,	defined by equation (14);
λ_{np},	eigenvalues of the characteristic equation (12);
ϕ,	angular coordinate [rad];
ω,	weighting function.

Dimensionless parameters

$Nu(x^+, \phi)$,	local Nusselt number;
Pe,	Peclet number;
Pr,	Prandtl number;
Re,	Reynolds number;
r^+,	radial coordinate defined by (3c);
x^+,	axial position defined by (3b);
θ,	local fluid temperature defined by (3a).

Subscripts

av,	refers to average value;
axial,	refers to axial direction;
fd,	evaluated far from the entrance;
w,	evaluated at wall condition;
ε,	evaluated at the tube entrance.

1. INTRODUCTION

RELATIVELY few papers appear in the literature dealing with laminar or turbulent heat transfer in a circular tube with variable circumferential wall heat flux or wall temperature [1–6] and none include the effect of axial fluid conduction.

The objective of the present work is to solve analytically the problem of heat transfer in a circular tube with an arbitrary circumferential wall heat flux for the case of a parabolic velocity profile with developing temperature profile including the effect of axial fluid conduction.

Heretofore, no general method has been available for obtaining a solution to this problem. Analytical results have been achieved in such generality and completeness that many of the previously-reported solutions in the heat-transfer literature for laminar tube flow are limiting cases of the present work.

The solution is expanded in a power series form that accounts for any arbitrary variation in heat flux around the circumference that can be expressed in terms of a Fourier expansion. Substitution of this series into the energy equation leads to an eigenvalue problem; the first 12 eigenvalues and corresponding eigenfunctions are obtained numerically. The resulting eigenfunctions are not orthogonal thus the power series expansion coefficients cannot be obtained by usual analytical schemes. A least squares method was employed to determine these coefficients.

2. FORMULATION OF PROBLEM

The problem to be treated is represented schematically in Fig. 1. We consider a steady, hydrodynamically developed, laminar, non-dissipative* viscous flow, of a conducting fluid with constant properties, flowing in a circular tube. The wall heat flux varies circumferentially according to the general function, $q(\phi)$, which is expressed in terms of a Fourier expansion.

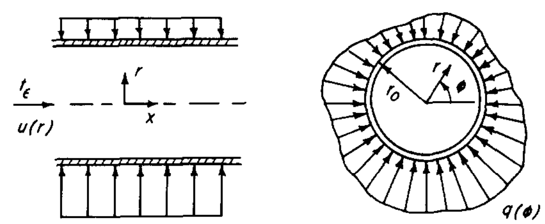

FIG. 1. Physical model and coordinate system.

The applicable form of the energy equation and boundary conditions are as follows:

$$\frac{2v}{\alpha}\left[1-\left(\frac{r}{r_0}\right)^2\right]\frac{\partial t}{\partial x} = \frac{\partial^2 t}{\partial r^2}+\frac{1}{r}\frac{\partial t}{\partial r}+\frac{1}{r^2}\frac{\partial^2 t}{\partial \phi^2}+\frac{\partial^2 t}{\partial x^2} \tag{1}$$

$$t(0, r, \phi) = t_\varepsilon \tag{2a}$$

$$t(\infty, r, \phi) = t_{fd} \tag{2b}$$

$$k\frac{\partial t}{\partial x}(x, r_0, \phi) = q(\phi) \tag{2c}$$

$$t(x, 0, \phi) = \text{finite} \tag{2d}$$

$$t(x, r, \phi) = t(x, r, \phi+2\pi) \tag{2e}$$

$$\frac{\partial t}{\partial \phi}(x, r, \phi) = \frac{\partial t}{\partial \phi}(x, r, \phi+2\pi). \tag{2f}$$

These equations are modified by introducing the following dimensionless variables

$$\theta = \frac{t-t_\varepsilon}{\bar{q}2r_0/\pi k} \tag{3a}$$

$$x^+ = \frac{x/r_0}{Re\,Pr} \tag{3b}$$

$$r^+ = \frac{r}{r_0} \tag{3c}$$

*This may be included and can be eliminated by a linear transformation of temperature [10].

where

$$\bar{q} = \int_0^{2\pi} q(\phi)\,d\phi \tag{3d}$$

with the requirement that $\bar{q} \neq 0$. Performing the necessary transformation we obtain the following forms for the energy equation and boundary conditions.

$$(1-r^{+2})\frac{\partial\theta}{\partial x^+} = \frac{\partial^2\theta}{\partial r^{+2}}+\frac{1}{r^+}\frac{\partial\theta}{\partial r^+}+\frac{1}{r^{+2}}\frac{\partial^2\theta}{\partial\phi^2}+\frac{1}{(Re\,Pr)^2}\frac{\partial^2\theta}{\partial x^{+2}} \tag{4}$$

$$\theta(0, r^+, \phi) = 0 \tag{5a}$$

$$\theta(\infty, r^+, \phi) = \theta_{fd}(\infty, r^+, \phi) \tag{5b}$$

$$\frac{\partial\theta}{\partial r^+}(x^+, 1, \phi) = \frac{q(\phi)}{\bar{q}}\frac{\pi}{2} \tag{5c}$$

$$\theta(x^+, 0, \phi) = \text{finite} \tag{5d}$$

$$\theta(x^+, r^+, \phi) = \theta(x^+, r^+, \phi+2\pi) \tag{5e}$$

$$\frac{\partial\theta}{\partial\phi}(x^+, r^+, \phi) = \frac{\partial\theta}{\partial\phi}(x^+, r^+, \phi+2\pi). \tag{5f}$$

We seek an exact solution, $\theta(x^+, r^+, \phi)$, satisfying equation (4) and the associated boundary conditions given by equations (5a–f). By experience with heat-conduction problems of similar form, a solution is sought of the form

$$\theta(x^+, r^+, \phi) = \theta_{fd}(x^+, r^+, \phi)+\theta^+(x^+, r^+, \phi) \tag{6}$$

in which $\theta_{fd}(x^+, r^+, \phi)$ is the asymptotic solution obtained far downstream where the temperature profile is fully developed, and θ^+ is the entry region solution.

Combining equations (4)–(6), and noting that for the case of a fully-developed temperature profile $\partial\theta_{fd}/\partial x^+$ = constant, we obtain differential equations and boundary conditions for the two regions as follows:

$$\frac{\partial^2\theta_{fd}}{\partial r^{+2}}+\frac{1}{r^+}\frac{\partial\theta_{fd}}{\partial r^+}+\frac{1}{r^{+2}}\frac{\partial^2\theta_{fd}}{\partial\phi^2} = (1-r^{+2})\frac{\partial\theta_{fd}}{\partial x^+} \tag{7a}$$

$$\frac{\partial\theta_{fd}}{\partial r^+}(x^+, 1, \phi) = \frac{q(\phi)}{\bar{q}}\frac{\pi}{2} \tag{7b}$$

$$\theta_{fd}(x^+, 0, \phi) = \text{finite} \tag{7c}$$

$$\theta_{fd}(x^+, r^+, \phi) = \theta_{fd}(x^+, r^+, \phi+2\pi) \tag{7d}$$

$$\frac{\partial\theta_{fd}}{\partial\phi}(x^+, r^+, \phi) = \frac{\partial\theta_{fd}}{\partial\phi}(x^+, r^+, \phi+2\pi) \tag{7e}$$

$$\frac{\partial^2\theta^+}{\partial r^{+2}}+\frac{1}{r^+}\frac{\partial\theta^+}{\partial r^+}+\frac{1}{r^{+2}}\frac{\partial^2\theta^+}{\partial\phi^2} = (1-r^{+2})\frac{\partial\theta^+}{\partial x^+}-\frac{1}{(Re\,Pr)^2}\frac{\partial^2\theta^+}{\partial x^{+2}} \tag{8a}$$

$$\theta^+(0, r^+, \phi) = -\theta_{fd}(0, r^+, \phi) \tag{8b}$$

$$\theta^+(\infty, r^+, \phi) = 0 \tag{8c}$$

$$\theta^+(x^+, 0, \phi) = \text{finite} \tag{8d}$$

$$\frac{\partial \theta^+}{\partial r^+}(x^+, 1, \phi) = 0 \qquad (8e)$$

$$\theta^+(x^+, r^+, \phi) = \theta^+(x^+, r^+, \phi + 2\pi) \qquad (8f)$$

$$\frac{\partial \theta^+}{\partial \phi}(x^+, r^+, \phi) = \frac{\partial \theta^+}{\partial \phi}(x^+, r^+, \phi + 2\pi) \qquad (8g)$$

3. DISCUSSION OF SOLUTION

Equations (7) are satisfied by a solution of the form

$$\theta_{fd}(x^+, r^+, \phi) = -\tfrac{7}{96} + x^+ + \frac{r^{+2}}{4} - \frac{r^{+4}}{16} + \sum_{n=1}^{\infty} r^{+n}(a_n \cos \phi + b_n \sin n\phi) - \frac{2}{\pi Pe^2}\int_0^{2\pi}\int_0^1 \frac{\partial \theta^+}{\partial x^+}(0, r^+, \phi) r^+ \, dr^+ \, d\phi \qquad (9)$$

where the coefficients, a_n and b_n may be determined from boundary condition (7b) with the usual Fourier analysis:

$$\begin{Bmatrix} a_n \\ b_n \end{Bmatrix} = \frac{1}{2n}\int_0^{2\pi} \frac{q(\phi)}{\bar{q}} \begin{Bmatrix} \cos n\phi \\ \sin n\phi \end{Bmatrix} d\phi. \qquad (10)$$

The integral appearing in equation (9) is the non-vanishing axial conduction term at the tube entrance. The need to include this term was discussed by Pirkle and Sigillito [7, 8].

Equations (8), applying to the entry region, are satisfied by

$$\theta^+(x^+, r^+, \phi) = \sum_{n=1}^{\infty}\sum_{p=0}^{\infty} e^{-\lambda_{np}^2 x^+} R_{np}(r^+) \times [a_{np}\cos p\phi + b_{np}\sin p\phi] \qquad (11)$$

where λ_{np} and R_{np} are the eigenvalues and eigenfunctions determined from

$$\frac{1}{r^+}\frac{d}{dr^+}\left(r^+ \frac{dR_{np}}{dr^+}\right) + \left[\lambda_{np}^2\left(1 - r^{+2} + \frac{\lambda_{np}^2}{Pe^2}\right) - \frac{p^2}{r^{+2}}\right] R_{np} = 0 \qquad (12)$$

with

$$R_{np}(0) = \text{finite}; \quad \frac{dR_{np}}{dr^+}(1) = 0.$$

Coefficients a_{np} and b_{np} in equation (11) are obtained using condition (8b) together with Fourier analysis in the variable ϕ. The expressions which result are

$$\sum_{n=1}^{\infty} a_{n0}\left[R_{n0}(r^+) + \frac{4\lambda_{n0}^2}{Pe^2}\int_0^1 R_{n0}(r^+) r^+ \, dr^+\right] = \frac{1}{2\pi}\int_0^{2\pi} \hat{\theta}_{fd}(r^+, \phi)\, d\phi \qquad (13a)$$

$$\sum_{n=1}^{\infty} \begin{Bmatrix} a_{np} \\ b_{np} \end{Bmatrix} R_{np}(r^+) = \frac{1}{\pi}\int_0^{2\pi} \hat{\theta}_{fd}(r^+, \phi) \begin{Bmatrix} \cos p\phi \\ \sin p\phi \end{Bmatrix} d\phi \qquad (13b)$$

with $p = 1, 2, 3, \ldots$ and

$$\hat{\theta}_{fd} = \tfrac{7}{96} - \frac{r^{+2}}{4} + \frac{r^{+4}}{16} - \sum_{n=1}^{\infty} r^{+n}(a_n \cos n\phi + b_n \sin n\phi). \qquad (14)$$

The coefficients a_{n0}, a_{np}, and b_{np} are now determined by expanding the functions of r^+ appearing on the right hand side of equation (13) in terms of the non-orthogonal eigenfunctions of the characteristic equation (12).

We now express equations (13a, b) in the general form

$$g(r^+) = \sum_{n=1}^{N} a_n R_n(r^+) \cong y(r^+) \qquad (15)$$

where $g(r^+)$ represents the integrals involving $\hat{\theta}_{fd}$ appearing on the right side of equations (13a, b), expressed in terms of a finite series of N terms, symbolized as $y(r^+)$.

In place of satisfying equation (15) at n points, it is often preferable to require that $y(r^+)$ and $g(r^+)$ agree as well as possible over a domain D of greater extent. This method involves the minimization of the integral of the square of the error in D. More generally, it is required that the squared error be multiplied by the weight function, $\omega(r^+)$, and that the product meets the condition

$$E(a_1, a_2, \ldots, a_N) = \int_0^1 \omega(r^+)\left[g(r^+) - \sum_{n=1}^{N} a_n R_n(r^+)\right]^2 dr^+ = \text{minimum}.$$

Minimizing with respect to each coefficient a_r, interchanging the order of differentiation and integration, and expanding we obtain

$$\sum_{n=1}^{N} a_n \int_0^1 \omega(r^+) R_n(r^+) R_r(r^+)\, dr^+ = \int_0^1 \omega(r^+) g(r^+) R_r(r^+)\, dr^+ \quad n = 1, 2, \ldots, N. \qquad (16)$$

For any arbitrary variation of circumferential wall heat flux, $g(r^+)$ is known. We choose $\omega(r^+) = r^+(1 - r^{+2})$, the weighting function when axial conduction is absent. Finally, the simultaneous equations are solved numerically to obtain the expansion coefficients.

At this point the complete solution is obtained by adding the fully-developed solution, equation (9), to the solution for the thermal entrance region, equation (11).

The mean fluid temperature over a given cross section may be determined by usual procedures, the expression which results is

$$\theta_{\text{mean}}(x^+) = x^+ + \frac{4}{Pe^2}\left\{\sum_{n=1}^{\infty} [1 - e^{-\lambda_{n0}^2 x}] \left[a_{n0}\lambda_{n0}^2 \int_0^1 R_{n0}(r^+) r^+ \, dr^+\right]\right\} \qquad 17)$$

Of particular practical interest are the values of wall temperature and of the local Nusselt member. The wall temperature is obtained by evaluating the complete solution at $r^+ = 1$ yielding

$$\frac{t_w - t_m}{\bar{q}2r_0/k\pi} = \tfrac{11}{96} + \sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi) + \frac{4}{Pe^2} \sum_{n=1}^{\infty} a_{n0} \lambda_{n0}^2 \, e^{-\lambda_{n0}^2 x^+} \int_0^1 R_{n0}(r^+) r^+ \, dr^+ + \sum_{n=1}^{\infty} a_{n0} R_{n0}(1) \, e^{-\lambda_{n0}^2 x^+} + \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} e^{-\lambda_{np}^2 x^+} R_{np}(1) \times [a_{np} \cos p\phi + b_{np} \sin p\phi]. \quad (18)$$

Local Nusselt number is now determined from this expression to yield the following expression

$$Nu(x^+, \phi) = \pi \frac{q(\phi)}{\bar{q}} \left[\tfrac{11}{96} + \sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi) + \frac{4}{Pe^2} \sum_{n=1}^{\infty} a_{n0} \lambda_{n0}^2 \, e^{-\lambda_{n0}^2 x^+} \int_0^1 R^{n0}(r^+) r^+ \, dr^+ + \sum_{n=1}^{\infty} a_{n0} R_{n0}(1) \, e^{-\lambda_{n0}^2 x^+} + \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} e^{-\lambda_{np}^2 x^+} R_{np}(1) \times [a_{np} \cos p\phi + b_{np} \sin p\phi] \right]^{-1} \quad (19)$$

4. SPECIAL EXAMPLES

As an illustrative case a cosine circumferential heat flux distribution of the form $q(\phi) = q_{av}(1 + b \cos \phi)$ is considered. Four special examples will be mentioned here. The interested reader may refer to Faghri [10], for details of these solutions including tables of eigenvalues and eigenfunctions.

Case 1. Letting $x^+ \to \infty$ equation (19) reduces to the asymptotic solution which is identical to that presented by Reynolds [1].

Case 2. With x^+ finite and $Pe \to \infty$ (no axial conduction) the Nusselt number expression reduces to the form given by Bhattacharya and Roy [6].

Case 3. For uniform wall heat flux ($b = 0$) and Pe finite the results of this analysis are identical to those presented by Hsu [9].

Case 4. We now wish to investigate the simultaneous effects of both circumferential wall heat flux variation and axial conduction.

Solving for a_n, b_n and $\bar{q}$ using equations (10) and (3d) and subsequently substituting into equations (18) and (19) the following results are obtained for wall temperature and for local Nusselt numbers

$$\frac{t_w - t_m}{q_{av} r_0/k} = \tfrac{11}{24} + \frac{4}{Pe^2} \sum_{n=1}^{\infty} \hat{a}_{n0} \lambda_{n0}^2 \, e^{-\lambda_{n0}^2 x^+} \times \int_0^1 R_{n0}(r^+) r^+ \, dr^+ + \sum_{n=1}^{\infty} \hat{a}_{n0} R_{n0}(1) \, e^{-\lambda_{n0}^2 x^+} + b \cos \phi \left[1 + \sum_{n=1}^{\infty} \hat{a}_{n1} R_{n1}(1) \, e^{-\lambda_{n1}^2 x^+} \right] \quad (20)$$

$$Nu(x^+, \phi) = 2(1 + b \cos \phi) \left\{ \tfrac{11}{24} + \sum_{n=1}^{\infty} \hat{a}_{n0} R_{n0}(1) \, e^{-\lambda_{n0}^2 x^+} + \frac{4}{Pe^2} \sum_{n=1}^{\infty} \hat{a}_{n0} \lambda_{n0}^2 \, e^{-\lambda_{n0}^2 x^+} \int_0^1 R_{n0}(r^+) r^+ \, dr^+ + b \cos \phi \left[1 + \sum_{n=1}^{\infty} \hat{a}_{n1} R^{n1}(1) \, e^{-\lambda_{n1}^2 x^+} \right] \right\}^{-1} \quad (21)$$

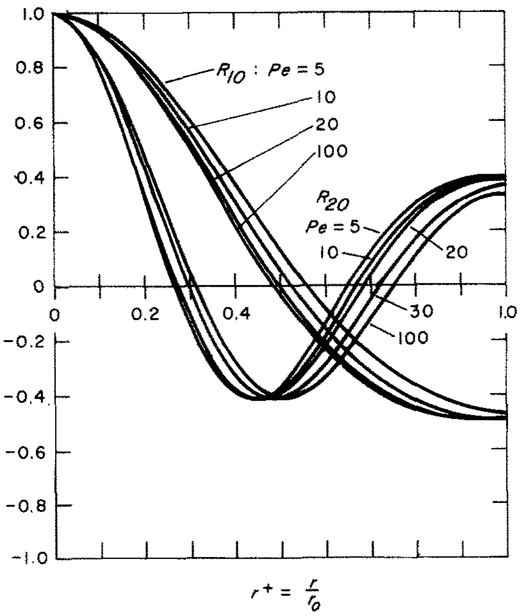

FIG. 2. First two eigenfunctions at $Pe = 5$, 10, 20, 30, 100; uniform wall heat flux ($p = 0$).

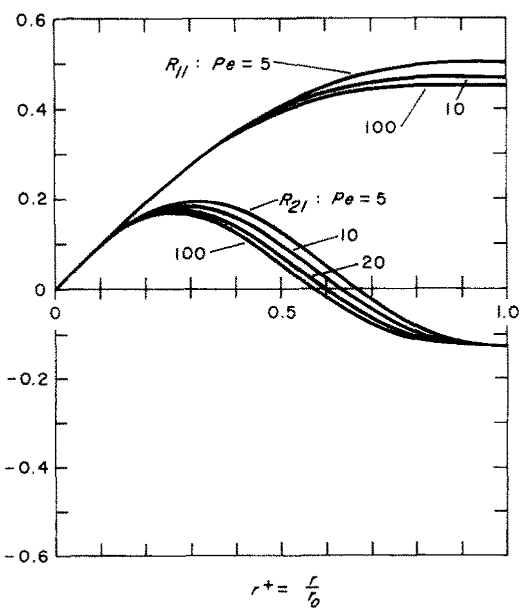

FIG. 3. First two eigenfunctions at $Pe = 5$, 10, 20, 30, 100; $p = 1$.

Table 1. Eigenvalues, eigenfunctions at tube wall, and expansion coefficients for $p = 1$ and Peclet numbers 5, 10, 20, 30, 50 and 100

n	λ_{n1}	$R_{n1}(1)$	$\hat{a}_{n1}$	$\int_0^1 r^+ R_{n1}(r^+)dr^+$
Pe = 5				
1	2.3395655	.5034665	-1.5191623E+00	.2179911
2	4.4798990	-.1308452	8.8401336E-01	.0005677
3	5.9496839	.0647956	-6.3939142E-01	.0038437
4	7.1376609	-.0401927	4.9323282E-01	.0006991
5	8.1589039	.0280160	-3.9963237E-01	.0006354
6	9.0676209	-.0209600	3.3390404E-01	.0002561
7	9.8940812	.0164431	-2.8342888E-01	.0002107
8	10.6571099	-.0133464	2.4148925E-01	.0001153
9	11.3693258	.0111140	-2.0410134E-01	.0000951
10	12.0396638	-.0094421	1.6840418E-01	.0000608
11	12.6747203	.0081514	-1.3148986E-01	.0000510
12	13.2795272	-.0071304	8.8061658E-02	.0000357
Pe = 10				
1	2.6695425	.4696647	-1.5523524E+00	.2076048
2	5.5982739	-.1309244	9.8436316E-01	-.0065759
3	7.7140852	.0661968	-7.7064946E-01	.0050551
4	9.4470416	-.0409682	6.0857278E-01	.0000932
5	10.9395496	.0284266	-4.9014303E-01	.0007459
6	12.2654614	-.0211918	4.0323442E-01	.0001594
7	13.4684788	.0165833	-3.3671212E-01	.0002257
8	14.5765360	-.0134362	2.8277229E-01	.0000897
9	15.6085876	.0111745	-2.3618108E-01	.0000976
10	16.5781218	-.0094844	1.9310344E-01	.0000515
11	17.4951109	.0081820	-1.4987862E-01	.0000513
12	18.3671596	-.0071531	1.0043446E-01	.0000316
Pe = 20				
1	2.8197826	.4536019	-1.5422761E+00	.2026325
2	6.4672994	-.1283298	1.0061598E+00	-.0123783
3	9.3694705	.0677853	-8.6554199E-01	.0071692
4	11.8350853	-.0426121	7.4194318E-01	-.0011056
5	13.9978558	.0295308	-6.2542009E-01	.0011225
6	15.9351062	-.0218906	5.2351322E-01	-.0000750
7	17.6980099	.0170330	-4.3769187E-01	.0002976
8	19.3224687	-.0137356	3.6507794E-01	.0000280
9	20.8344119	.0113810	-3.0216865E-01	.0001146
10	22.2530309	-.0096315	2.4497449E-01	.0000300
11	23.5928677	.0082898	-1.8918506E-01	.0000560
12	24.8651860	-.0072340	1.2758978E-01	.0000225
Pe = 30				
1	2.8545778	.4498334	-1.5022573	.2014624
2	6.7747463	-.1263716	1.1175630	-.0142855
3	10.1060500	.0678991	-1.3334928	.0083909
4	13.0364491	-.0435948	1.5110602	-.0020467
5	15.6603235	.0305244	-1.5220368	.0015572
6	18.0423447	-.0226581	1.4107798	-.0003433
7	20.2282494	.0175836	-1.2503466	.0004125
8	22.2524521	-.0141264	1.0876408	-.0000504
9	24.1416343	.0116617	- .9429579	.0001471
10	25.9167186	-.0098370	.8214723	.0000031
11	27.5942514	.0084434	- .7218267	.0000664
12	29.1874493	-.0073511	.6406117	.0000114
Pe = 50				
1	2.8735619	.4477705	-1.5009765	.2008212
2	6.9802511	-.1246824	1.0000705	-.0154775
3	10.6992998	.0672620	-1.1034367	.0094188
4	14.1388678	-.0440567	1.3498381	-.0031129
5	17.3263081	.0315592	-1.5751142	.0022441
6	20.2923130	-.0237911	1.6938552	-.0008407
7	23.0652471	.0185858	-1.6963028	.0006903
8	25.6690302	-.0149384	1.6138519	-.0002397
9	28.1238534	.0122971	-1.4854820	.0002480
10	30.4470629	-.0103296	1.3414231	-.0000694
11	32.6536637	.0088263	-1.2001143	.0001041
12	34.7566170	-.0076514	1.0707484	-.0000187
Pe = 100				
1	2.8818326	.4468702	-1.5003515	.2005413
2	7.0818961	-.1237271	.9272440	-.0160374
3	11.0437856	.0664633	- .8656810	.0099990
4	14.8831962	-.0437110	.9498700	-.0038894
5	18.6004398	.0317845	-1.1274227	.0029304
6	22.1915814	-.0245038	1.3542831	-.0014884
7	25.6562355	.0196010	-1.5852410	.0011865
8	28.9973910	-.0160734	1.7819754	-.0006439
9	32.2200108	.0134205	-1.9201843	.0005328
10	35.3298632	-.0113665	1.9913475	-.0002883
11	38.3328769	.0097447	-1.9996337	.0002519
12	41.2348869	-.0084454	1.9567307	-.0001313

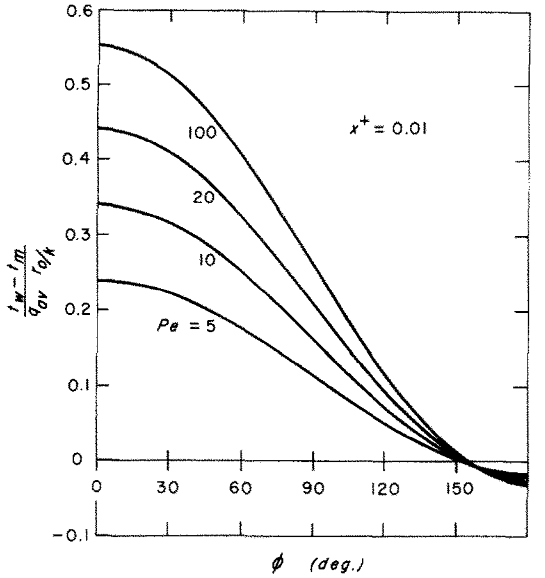

FIG. 4. Dimensionless wall temperature vs ϕ for wall heat flux variation given by $q(\phi) = q_{av}(1+\cos\phi)$; entrance region ($x^+ = 0.01$).

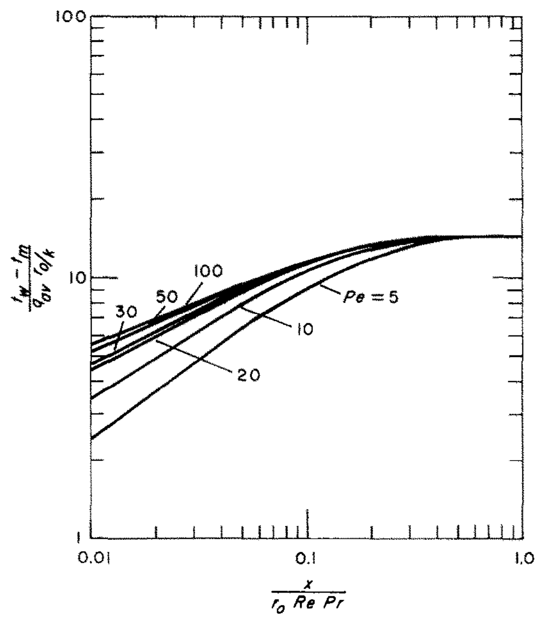

FIG. 5. Dimensionless wall temperature in entrance region at $Pe = 5, 10, 20, 30, 50, 100$; at location of maximum wall heat flux ($\phi = 0$); $q(\phi) = q_{av}(1+\cos\phi)$.

The coefficients, $\hat{a}_{n0}$ and $\hat{a}_{n1}$, are obtained using the least-squares procedures as previously discussed. The validity of the least-squares approximation, as related to this case, is demonstrated by Faghri [10].

Values of $\hat{a}_{n0}$, λ_{n0}, and R_{n0} have been presented by Hsu [9]; they are also given by Faghri [10], who evaluated $\hat{a}_{n0}$ using the least-squares technique. Values of λ_{n1}, R_{n1}, and $\hat{a}_{n1}$ are listed in Table 1 for a range in Pe between 5 and 100. Additional tables of computed values for $p = 2$ were evaluated by Faghri [10].

Variations in the first two eigenfunctions are shown in Figs. 2 and 3 for several Peclet numbers.

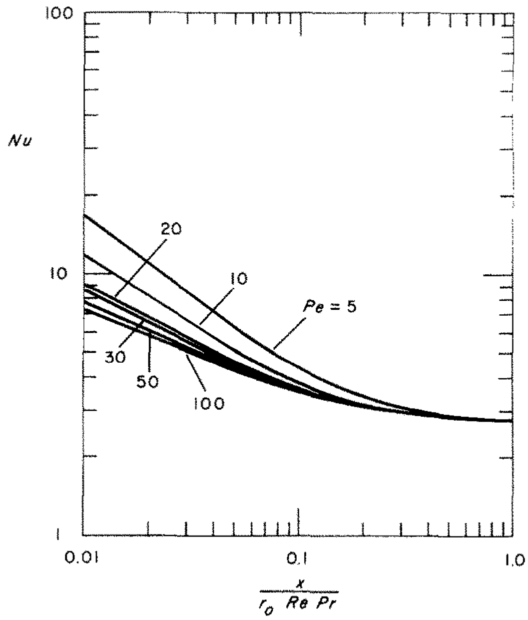

FIG. 6. Nusselt number variation in entrance region at $Pe = 5, 10, 20, 30, 50, 100$; at location of maximum wall heat flux ($\phi = 0$); $q(\phi) = q_{av}(1+\cos\phi)$.

Having evaluated eigenvalues, eigenfunctions, and expansion coefficients we are now able to determine values for temperature and the Nusselt number at locations of interest. Figure 4 shows the wall temperature variation as a function of circumferential position at $x^+ = 0.01$, which is illustrative of conditions in the entrance region.

Dimensionless wall temperature and Nusselt numbers are illustrated in Figs. 5 and 6, respectively. In each figure the parameters are evaluated at $\phi = 0$, where heat flux is maximum.

The effect of axial conduction on heat transfer may be inferred from these figures. For values of Pe greater than 100 axial conduction is seen to be negligible. At Peclet numbers less than 100, heat transfer is influenced significantly by axial conduction effects within the thermal entry region, $x^+ \simeq 0.5$. There is also a pronounced effect on circumferential temperature distribution.

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ANALYSE DU TRANSFERT THERMIQUE, COMPTE TENU DE LA CONDUCTION AXIALE DU FLUIDE- POUR UN ECOULEMENT LAMINAIRE DANS UN TUBE AVEC VARIATION CIRCONFERENTIELLE QUELCONQUE DU FLUX PARIETAL

Résumé—On présente la solution exacte du probléme du transfert thermique à l'entrée d'un tube à section circulaire avec flux pariétal à distribution circonférentielle quelconque et en tenant compte de la conduction axiale dans le fluide. La solution est donné sous la forme d'une série puissance; les 12 premières valeurs propres et fonctions propres sont obtenues numériquement. La méthode des moindres carrés est employée pour déterminer les coefficients du développement en série pour le système non orthogonal. On donne un résultat simple pour la variation du flux thermique selon $q = q_{av}(1+\cos\phi)$, qui illustre l'influence simultanée de la variation circonférentielle du flux pariétal et de la conduction axiale dans le fluide, sur le nombre de Nusselt.

BERECHNUNG DES WÄRMEÜBERGANGS BEI LAMINARER ROHRSTRÖMUNG MIT DEFINIERTER VERÄNDERUNG DES WÄRMESTROMS AN DER WAND UNTER BERÜCKSICHTIGUNG DER WÄRMELEITUNG DES FLUIDS IN AXIALER RICHTUNG

Zusammenfassung—Es wird eine Berechnung angegeben für die exakte Lösung des Problems des Wärmeübergangs im thermischen Einlaufgebiet eines Kreisrohres mit definiertem Wärmestrom am Wandumfang, einschließlich des Effekts der Wärmeleitung des Fluids in axialer Richtung. Die Lösung wird in Form einer Potenzreihe dargestellt; die ersten 12 Eigenwerte und Eigenfunktionen wurden numerisch erhalten. Mit der Methode der kleinsten Fehlerquadrate werden die Koeffizienten der Reihenentwicklung für ein nicht-orthogonales System bestimmt. Es wird eine einfache Gleichung für die Veränderung des Wärmestroms in der Form $q = q_{av}(1+\cos\theta)$ angegeben, welche den gleichzeitigen Einfluß der Veränderung des Wärmestroms am Wandumfang und der axialen Wärmeleitung in der Flüssigkeit auf die Wandtemperatur und die Nußelt-Zahl beschreibt.

ИССЛЕДОВАНИЕ ТЕПЛООБМЕНА ПРИ ЛАМИНАРНОМ ТЕЧЕНИИ В ТРУБЕ В СЛУЧАЕ ПРОИЗВОЛЬНОГО ИЗМЕНЕНИЯ ПО ОКРУЖНОСТИ ТРУБЫ ТЕПЛОВОГО ПОТОКА НА СТЕНКЕ С УЧЕТОМ ТЕПЛОПРОВОДНОСТИ ЖИДКОСТИ НА ОСИ

Аннотация — Анализируется точное решение задачи переноса тепла в начальном участке круглой трубы при произвольном изменении по окружности теплового потока на стенке с учетом влияния осевой теплопроводности жидкости. Решение представлено в виде степенного ряда; численно получены первые 12 собственных значений и функций. Для определения коэффициентов разложения неортогональной системы использовался метод наименьших квадратов. Изменение теплового потока представлено в виде простого отношения $q = q_{av}(1+\cos\phi)$, которое учитывает одновременное влияние изменения по окружности теплового потока на стенке и осевой теплопроводности жидкости на температуру стенки и число Нуссельта.